UDC 534

ON THE GENERATION OF SETS OF SUBHARMONIC MODES IN A PIECEWISE-CONTINUOUS SYSTEM

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For a nonautonomous system with one degree of freedom we have obtained the conditions for the generation of complex subharmonic oscillations and we have shown the possibility of the generation of whole sets of unstable subharmonic modes by violation of conditions for existence of periodic motion connected with the change in the sequence of passage of the phase trajectory through regions of piecewise-continuity. The periodic mode of the motion of a piecewise-continuous system is characterized by a specific sequence of passage of the phase trajectory through regions of piecewise-continuity. Every violation of this sequence of joining ["sewing together"] the separate pieces of trajectories signifies a violation of conditions for the existence of periodic motion of a given type and corresponds to a certain C-bifurcation. In the simplest case, since the system parameters vary, it is possible to have here either a transition of one type of mode into another or a merging of modes of two different types and their subsequent vanishing. The more complicated case of the doubling of the period of oscillation at a C-bifurcation was analyzed in [1]. There are no fundamental difficulties in the study of the case of generation of a subharmonic mode of order $1 / n$, whose n rotations of phase trajectory are joined in a definite manner from two types of trajectories of the modes participating in the bifurcation. However, the awkwardness of the expression for the conditions of existence of such mode makes the solution of the problem in the general case unsuitable for the investigation of a real system.

In the present paper we investigate new cases of C-bifurcation on the example of the forced oscillations of a linear dissipative system with one degree of freedom. The bifurcations are connected with the setting of a motion limiter. We have obtained the conditions for the generation of complex subharmonic oscillations and have shown the possibility of generation of whole sets of unstable subharmonic modes. The number of these modes increases as the period of the external excitation decreases, while the structure of the parameter space partitioning into regions of their existence becomes all the more "fine".

The investigation of the periodic motions of piecewise-continuous systems is often met with insurmountable difficulties associated with the determination of the stability region for a mode of a given type, as well as with the impossibility of preassigning the types of motion which are realized in specified regions of the parameter space. Therefore, the detection of cases of unstable mode generation and the investigation of regions of their existence is of particular practical importance, if they permit the disclosure of dangerous boundaries of the region of stable modes [2]. In this paper we calculate a dangerous boundary in the parameter space, corresponding to the vanishing of an unstable subharmonic mode, because of its merging with a stable mode of the same type. It turns out that the ratio of the "amplitude" of the dangerous mode to the amplitude of the forced linear oscillations of the system with the same parameter values grows strongly with the increase of the order of the subharmonic mode. Thus, a subharmonic mode can manifest itself the stronger the finer is the structure of the parameter space partitioning.

1. Let us examine the C-bifurcations of the forced oscillations of a linear dissipative system with one degree of freedom when it collides with a fixed motion limiter. The equations of motion for the system written in dimensionless form, are

$$x^{\cdot\cdot} + 2\lambda x^{\cdot} + x = P(\tau), \qquad x < d \tag{1.1}$$

$$x_+^{\cdot} = -Rx_-^{\cdot}, \qquad x = d \tag{1.2}$$

Here $P(\tau)$ is a T-periodic time function, the coefficient λ characterizes the viscous friction $(0 \leqslant \lambda < 1)$, and R is the velocity restitution coefficient under impact $(0 \leqslant R < 1)$. We denote the considered motions by $\Gamma(n, k)$, where k is the number of impacts (1.2) within a period equal to nT $(n = 1, 2, \ldots)$.

Let $p(\tau)$ be a particular solution of Eq. (1.1), corresponding to the steady-state forced oscillations of the linear system. Then the general solution of the linear equation (1.1) can be written as an equation of the point transformation corresponding to the segment of the phase trajectory between the point $M_i(x_i, x_i^{\cdot}, \tau_i)$ and the point $M_j(x_j, x_j^{\cdot}, \tau_j)$

$$x_j = p_j + e^{-\lambda\tau_{ij}}\left[(x_i - p_i)\left(\frac{\lambda}{\delta}\sin\delta\tau_{ij} + \cos\delta\tau_{ij}\right) + \frac{x_i^{\cdot} - p_i^{\cdot}}{\delta}\sin\delta\tau_{ij}\right] \tag{1.3}$$

$$x_j^{\cdot} = p_j^{\cdot} + e^{-\lambda\tau_{ij}}\left[(x_i^{\cdot} - p_i^{\cdot})\left(\cos\delta\tau_{ij} - \frac{\lambda}{\delta}\sin\delta\tau_{ij}\right) - \frac{x_i - p_i}{\delta}\sin\delta\tau_{ij}\right]$$

$$\tau_{ij} = \tau_j - \tau_i, \qquad \delta = \sqrt{1 - \lambda^2}$$

The contact of the limiter surface with the phase trajectory corresponds to the case of bifurcation considered here. The equation of this singular trajectory is

$$x(\tau) = p(\tau), \qquad x^{\cdot}(\tau) = p^{\cdot}(\tau), \qquad p(\tau)_{\max} = d_* \tag{1.4}$$

It is obvious that the equation for $\Gamma(n, 0)$ coincides with Eq. (1.4) for $\Gamma(1,0)$, while the trajectory of $\Gamma(n, 0)$ can be treated as the limit case of the n-reverse trajectory of $\Gamma(n, 1)$ as the pre-impact velocity tends to zero. With such an approach we can use the results obtained in [1] for answering the question on the nature of the bifurcations of the periodic modes $\Gamma(n, 0)$ and $\Gamma(n, 1)$ as they merge.

Let $\chi_{n0}(z)$ and $\chi_{n1}(z)$ be the characteristic polynomials corresponding to $\Gamma(n, 0)$ and $\Gamma(n, 1)$ in the limit case of merging of their trajectories. Then, since the parameter d varies, one motion passes into the other, if the condition

$$\chi_{n0}(+1)\ \chi_{n1}(+1) > 0 \tag{1.5}$$

is satisfied; the motions $\Gamma(n, 0)$ and $\Gamma(n, 1)$ vanish after merging, if the condition

$$\chi_{n0}(+1)\ \chi_{n1}(+1) < 0 \tag{1.6}$$

is satisfied; a doubled-period motion $\Gamma(2n, 1)$ is generated, if the condition

$$\chi_{n0}(-1)\,\chi_{n1}(-1) < 0 \qquad (1.7)$$

is satisfied.

To obtain $\chi_{n0}(z)$ we analyze the point transformation $M_j\,(M_i)$ of the halfplane $x_i = 0,\ x_i^{\cdot} < 0$ into itself, generated by n reversals of phase trajectory (1.3). Here $x_j^{\cdot} = x_i^{\cdot} = p^{\cdot}(\tau_i),\ \tau_{ij} = nT$, correspond to the fixed point of the transformation. After the usual procedure of the variation of (1.3) with respect to the variables $x_i^{\cdot},\ x_j^{\cdot},\ \tau_i,\ \tau_j$ and the substitutions $\delta x_j^{\cdot} = z\delta x_i^{\cdot},\ \delta\tau_j = z\delta\tau_i$, we arrive at the expression

$$\chi_{n0}(z) = x_i^{\cdot}\,[z^2 - 2ze^{-\lambda nT}\cos(nT\delta) + e^{-2\lambda nT}], \qquad n = 1, 2, 3, \ldots \qquad (1.8)$$

To obtain $\chi_{n1}(z)$ we analyze the transformation $M_l\,(M_i)$ of the halfplane $x_i = 0$, $x_i^{\cdot} < 0$ into itself, generated by n reversals of the phase trajectory, which consists of successive transformation of $M_j\,(M_i)$ in the plane $x_j = d$ in accordance with Eqs. (1.3), the transformation $M_k\,(M_j)$ of the impact interaction (1.2), and the further transformation $M_l\,(M_k)$ again in accordance with Eqs. (1.3) (Fig. 1). In the limiting case of C-bifurcation the characteristic polynomial assumes the following form:

$$\chi_{n1}(z) = -zx_i^{\cdot}\delta^{-1}(1 + R)\,x_j^{\cdot\cdot}e^{-\lambda nT}\sin(nT\delta), \qquad n = 1, 2, 3, \ldots \qquad (1.9)$$

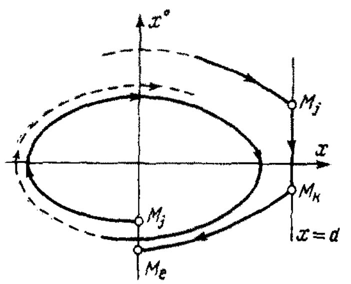

Fig. 1

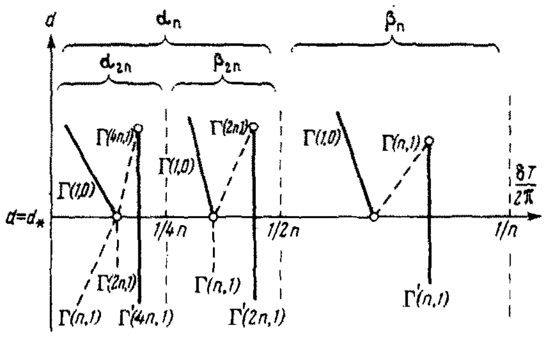

Fig. 2

The acceleration $x_j^{\cdot\cdot} < 0$ because the function $x(\tau)$ attains its maximum for $\tau = \tau_j$. Consequently, in the considered case conditions (1.5) – (1.7) can be written as follows:

a) the condition for the transition of mode $\Gamma(n, 0)$ into mode $\Gamma(n, 1)$ and the coincident condition of the generation of the subharmonic mode $\Gamma(2n, 1)$

$$\sin(nT\delta) > 0 \qquad (1.10)$$

b) the condition for the vanishing of modes $\Gamma(n, 0)$ and $\Gamma(n, 1)$ as a result of their merging

$$\sin(nT\delta) < 0 \qquad (1.11)$$

Here the characteristic polynomial (1.9) implies the instability of all generated or vanishing modes of type $\Gamma(n, 1)$.

For simplicity we restrict ourselves below to the examination of two sequences of intervals of variation of the period of the external force: the sequences α_n defined by the inequality $0 < T / 2\pi < 1 / 2n\delta$ and the sequences β_n defined by $1 / 2n\delta < T / 2\pi < 1 / n\delta$. According to (1.11) the stable mode $\Gamma(n, 0)$ merges with the unstable

$\Gamma\ (n,\ 1)$ in intervals β_n and both modes vanish with the subsequent decrease of parameter d. According to (1.10), in intervals α_n the transition of the stable mode $\Gamma\ (n,\ 0)$ to the unstable $\Gamma\ (n,\ 1)$ is accompanied either by merging with the unstable mode $\Gamma\ (2n,\ 1)$ or by the generation of an unstable mode $\Gamma\ (2n,\ 1)$. The generation of $\Gamma\ (2n,\ 1)$ obtains if $T\ /\ 2\pi$ falls into the first half of interval α_n, since this is equivalent to $T\ /\ 2\pi \in \alpha_{2n}$. The merging with $\Gamma\ (2n,\ 1)$ obtains if $T\ /\ 2\pi$ falls into the second half of interval α_n, since this is equivalent to $T\ /\ 2\pi \in \beta_{2n}$ (Fig. 2). If the period of the external excitation decreases, the value $T\ /\ 2\pi$ will belong to an ever larger number of intervals α_n, β_n, and, consequently, an ever more complex set of subharmonic modes with C-bifurcations of the forced oscillations of the linear system $\Gamma\ (1,\ 0) \equiv \Gamma\ (n,\ 0)\ (n = 2,\ 3,\ 4,\ \ldots)$ will be generated.

Thus, for example, the merging of $\Gamma\ (1,\ 0)$ with the subharmonic modes $\Gamma\ (2,\ 1)$ and $\Gamma\ (3,\ 1)$ and the generation of the mode $\Gamma\ (1,1)$ take place in the interval $T\delta\ /\ 2\pi \in (^1/_4,\ ^1/_3)$; in the interval $T\delta\ /\ 2\pi \in (^1/_5,\ ^1/_4)$, when the parameter d decreases, the mode $\Gamma\ (1,\ 0)$ is merged with the modes $\Gamma\ (3,\ 1)$, $\Gamma\ (4,\ 1)$ and further modes $\Gamma\ (1,1)$ and $\Gamma\ (2,\ 1)$ etc. are generated.

Thus, from the fact that unstable periodic modes of type $\Gamma\ (n,\ 1)$ exist in the intervals $\beta_1,\ \beta_3,\ \beta_5,\ \ldots$ for $d > d_*$ follows the existence in the neighborhood of $d = d_*$ of a whole sequence of unstable modes $\Gamma\ (2n,\ 1)$, $\Gamma\ (4n,\ 1)$, $\Gamma\ (8n,\ 1), \ldots$ in the corresponding intervals of variation of the external force period. Here, when T decreases, the doubled-period mode $\Gamma\ (2n,\ 1)$ appears in the region $d > d_*$ simultaneously with a reorientation of the existence region of mode $\Gamma\ (n,\ 1)$ relative to the axis $d = d_*$ (Fig. 2) . The existence of mode $\Gamma\ (1,\ 1)$ for $d > d_*$ was proved earlier in [1]. We will consider the question of the existence of more complex modes of type $\Gamma\ (n,\ k)$ in the neighborhood of $d = d_*$.

2. A periodic mode $\Gamma\ (n,\ k)$ during which k impact interactions with the limiter take place, is assumed to be close to the periodic mode of forced oscillations of the linear system in the sense that these modes merge when $d \to d_*$. Let us reduce the problem to an investigation of point transformations of the halfplane $x = d,\ x^{\cdot} < 0$ into itself. The phase trajectories of impact interactions are located precisely in this plane. A specific sequence of points

$$\ldots,\ M_0\ (x_0^{\cdot},\ \tau_0),\ M_1\ (x_1^{\cdot},\ \tau_1),\ \ldots,\ M_k\ (x_k^{\cdot},\ \tau_k),\ \ldots$$

of the transformation corresponds to the considered periodic modes.

Let us simplify somewhat the problem by assuming the external force to be harmonic $P\ (\tau) = \cos\omega\tau$, and the viscous friction coefficient $\lambda = 0$, i.e. by keeping the system dissipative only at the expense of not fully elastic impacts. We then obtain the equations of the point transformations from Eqs. (1.3) with $\lambda = 0,\ \delta = 1,\ p\ (\tau) = (1 - \omega^2)^{-1}\cos\omega\tau$ and from Eq. (1.2)

$$\begin{aligned} x_{i+1} = d = a\cos\omega\tau_{i+1} + (x_i^{\cdot} + a\omega\sin\omega\tau_i)\sin(\tau_{i+1} - \tau_i) + \\ (d - a\cos\omega\tau_i)\cos(\tau_{i+1} - \tau_i) \end{aligned} \qquad (2.1)$$

$$\begin{aligned} -\frac{\dot{x}_{i+1}}{R} = -a\omega\sin\omega\tau_{i+1} + (x_i^{\cdot} + a\omega\sin\omega\tau_i)\cos(\tau_{i+1} - \tau_i) - \\ (d - a\cos\omega\tau_i)\sin(\tau_{i+1} - \tau_i) \end{aligned}$$

$$a = (1 - \omega^2)^{-1}$$

The considered periodic mode is characterized by the following sequence of fixed points:

$$(x_0^{\cdot}, \tau_0), \quad (x_1^{\cdot}, \tau_1), \ldots, \quad (x_k^{\cdot} = x_0^{\cdot}, \tau_k = \tau_0 + 2\pi n / \omega) \qquad (2.2)$$

In the limiting case of C-bifurcation this mode degenerates into the forced oscillation mode of the linear system and, consequently,

$$x_i^{\cdot *} = 0, \quad \sin \omega\tau_0^* = 0, \quad a \cos \omega\tau_0^* = d_* \qquad (2.3)$$

$$\tau_{i+1}^* - \tau_i^* = m_i T, \quad \sum_0^{k-1} m_i = n, \quad i = 0, 1, \ldots, k-1$$

Instead of the variables τ_i we introduce the new variables $\varepsilon_i = \tau_i - \tau_i^*$, and define the position of the limiter by the small parameter $\mu = d - d_*$. In the neighborhood of the degenerate trajectory the coordinates of the points of the transformation are $|x_i^{\cdot}| \ll 1$, $|\varepsilon_i| \ll 1$, which permits us to represent the equations of the point transformations (2.1) in the linearized form

$$(x_i^{\cdot} + d_*\omega^2\varepsilon_i) \sin m_i T = \mu (1 - \cos m_i T), \; i = 0, 1, \ldots, k-1 \qquad (2.4)$$

$$(x_i^{\cdot} + d_*\omega^2\varepsilon_i) \cos m_i T - d_*\omega^2\varepsilon_{i+1} + x_{i+1}^{\cdot}/R = \mu \sin m_i T$$

Together with the periodicity conditions

$$\varepsilon_k = \varepsilon_0, \quad x_k^{\cdot} = x_0^{\cdot} \qquad (2.5)$$

system (2.4) yields the following coordinates for the fixed points of the complex mode:

$$x_i^{\cdot} = \frac{\mu R}{1+R}\left(\operatorname{tg}\frac{m_i T}{2} + \operatorname{tg}\frac{m_{i-1}T}{2}\right), \qquad i = 0, 1, \ldots, k-1 \qquad (2.6)$$

$$d_*\omega^2(1+R)\varepsilon_i = \mu\left(\operatorname{tg}\frac{m_i T}{2} - R\operatorname{tg}\frac{m_{i-1}T}{2}\right)$$

Let us further obtain the conditions of existence for a mode of the specified type, i.e. conditions for the absence of additional impacts in the time interval between the specified impact interactions. In the interval $\tau_i < \tau < \tau_{i+1}$, in the neighborhood of the degeneration boundary the function

$$x(\tau) = a \cos \omega\tau + (d - a \cos \omega\tau_i) \cos (\tau - \tau_i) + (x_i^{\cdot} + a\omega \sin \omega\tau_i) \sin (\tau - \tau_i)$$

has $m_i - 1$ maxima reached at the instants τ_{ij} $(j = 1, 2, \ldots, m_i - 1)$. The equation for determining τ_{ij}

$$x^{\cdot}(\tau_{ij}) = 0$$

can also be represented in a linearized form, if we pass to the variables

$$\gamma_{ij} = \tau_{ij} - \tau_i^* - \varepsilon_i - jT, \quad j = 1, 2, \ldots, m_i - 1$$

After appropriate transformations we obtain in linear approximation the following values of function $x(\tau)$ at points of maximum:

$$x(\tau_{ij}) = d_* + \mu \cos (jT) + \mu \sin (jT) \operatorname{tg}\frac{m_i T}{2} \qquad (2.7)$$

For modes of the specified type to exist the values determined by (2.7) must not reach d. Consequently, the sought conditions of existence can be written as

$$\mu\left[1-\cos(jT)-\sin(jT)\,\mathrm{tg}\,\frac{m_iT}{2}\right]>0, \qquad j=1,\ldots,m_i-1 \tag{2.8}$$

Having supplemented these conditions by the requirement $x_i^{\cdot}<0$, implied by the method of construction of point transformations (2.1), we obtain from (2.6) and (2.8), after elementary trigonometric transformations, a system of inequalities which are the necessary conditions of existence in a neighborhood of a C-bifurcated degeneration of the considered periodic modes

$$\mu\sin\frac{\pi(m_i+m_{i-1})}{\omega}\cos\frac{\pi m_{i-1}}{\omega}\cos\frac{\pi m_i}{\omega}<0 \tag{2.9}$$

$$\mu\sin\frac{\pi(m_i-j)}{\omega}\cos\frac{\pi m_i}{\omega}\sin\frac{\pi j}{\omega}<0$$

$$i=0,1,\ldots,k-1, \qquad j=1,2,\ldots,m_i-1$$

The periodic sequence of values $x_i^{\cdot}$, τ_i, m_i must evidently not decompose into simpler periodic sequences.

Example 1. By setting $m_0=n$, $k=1$ from (2.9) we obtain the existence condition for an n-fold one-impact periodic mode $\Gamma(n, 1)$

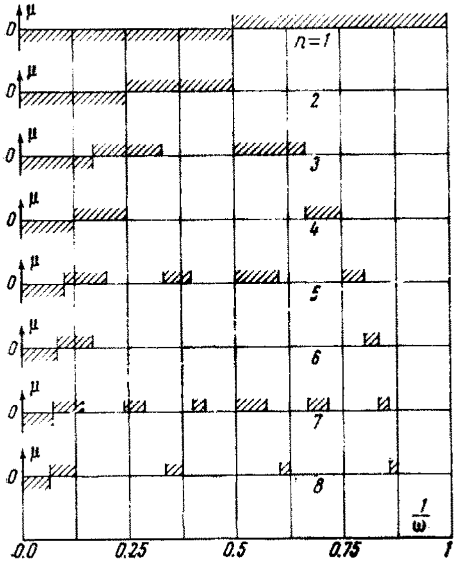

Fig. 3

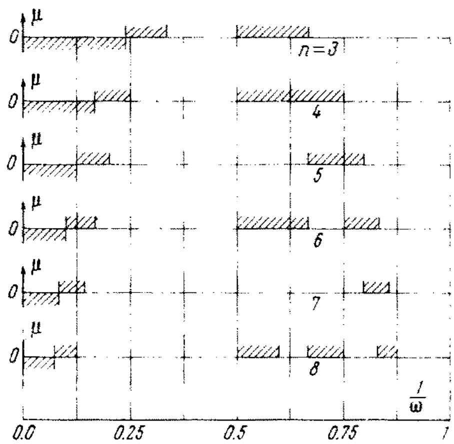

Fig. 4

$$\mu\sin\frac{2\pi n}{\omega}<0, \quad \mu\sin\frac{\pi(n-j)}{\omega}\sin\frac{\pi j}{\omega}\cos\frac{\pi n}{\omega}<0, \quad j=1,\ldots,n-1 \tag{2.10}$$

The results of the analysis of inequalities (2.10) in the frequency interval $0<1/\omega<1$ for $n=1, 2, \ldots, 8$ are presented in Fig. 3. The hatching shows the regions of existence for modes of the corresponding multiplicity. In all cases only the region corresponding to the highest frequency lies below the axis $\mu=0$. All other regions correspond to $\mu>0$, i.e. to $d>d_*$. Consequently, in these cases unstable subharmonic oscillations of order n with impacts exist side by side with the stable linear oscillations.

If we compare the outlined regions of existence for even values of n with the regions defined by the doubled-period conditions (1.10), we conclude that only in the two left-hand regions the generation of even harmonics is due to the doubling of the period at a

C-bifurcation. The other cases correspond, obviously, to a more complex case of bifurcation.

Example 2. Let us examine an n-fold two-impact mode. Since similar modes can now be of several types, we restrict our attention to the case when the interval between impacts approximately equals the period of the external force. In that case $m_0 = 1$, $m_1 = n - 1$, $k = 2$, and conditions of existence (2.9) assume the form

$$\mu \sin \frac{\pi n}{\omega} \cos \frac{\pi}{\omega} \cos \frac{\pi (n-1)}{\omega} < 0 \qquad (2.11)$$

$$\mu \sin \frac{\pi (n-1-j)}{\omega} \cos \frac{\pi (n-1)}{\omega} \sin \frac{\pi j}{\omega} < 0, \qquad j = 1, 2, \ldots, n-2$$

The regions of existence determined by the analysis of conditions (2.11) are shown in Fig. 4. As in the preceding case all regions except those corresponding to the highest frequency are arranged for $\mu > 0$, and, consequently, the corresponding unstable modes exist side by side with the mode of forced linear oscillations.

3. The described analysis which revealed the generation of whole sets of unstable modes of type $\Gamma(n, k)$ from the simplest mode of linear oscillations when the displacement limiter is reached, leads to the following problem: to ascertain the reasons and the values of parameters for which these modes vanish. A reasonably complete solution of such problem is not possible. However, the finding of the so-called dangerous boundaries of the stability regions [2] is of great practical value. In relation to the investigated system this corresponds to the vanishing of unstable modes by merging with the stable modes in the region of parameters $d > d_*$. It should be noted that the question of the existence of stable nonlinear modes side by side with the forced linear oscillations in an oscillatory system with limiters is of independent interest and had been repeatedly investigated (for example, see [3 – 8]).

It can be shown that the "most dangerous" stable nonlinear mode $\Gamma'(n, 1)$ necessarily exists in the neighborhood of the angle of bifurcation $\omega = 2n$, $d = d_*$, and it vanishes by merging with an unstable mode of the same type as $\Gamma(n, 1)$ as the parameter d increases (Fig. 2). The corresponding value d_0 should be treated as the minimally admissible position of the limiter. If $d < d_0$, then under the action of some random factors it is possible for the dynamic system to pass from the mode $\Gamma(1, 0)$ with oscillation amplitude d_* to the stable subharmonic mode $\Gamma'(n, 1)$ whose oscillation "amplitude" can considerably exceed d_*.

Let us find the expression for d_0 in the case of a harmonic perturbation $P(\tau) = \cos \omega \tau$ taking into account the viscous friction coefficient λ. For this we consider the transformation of the plane $x_0 = d$ into itself, generated by a segment of phase trajectory (1.3) and a single impact interaction (1.2). Setting $x_1 = x_0 = d$, $x_1^{\cdot} = x_0^{\cdot}$, $\tau_1 = \tau_0 + 2\pi n / \omega$, we arrive at the following equations relative to the coordinates of the fixed point of the transformation:

$$d = \frac{a (\cos \omega\tau_0 + 2\omega a \lambda \sin \omega\tau_0)}{1 + (2a\omega\lambda)^2} + \frac{x_0^{\cdot} \rho \sin \theta\delta}{\delta (\operatorname{ch} \theta\lambda - \cos \theta\delta)} \qquad (3.1)$$

$$\frac{x_0^{\cdot}}{R} = \frac{a\omega (\sin \omega\tau_0 - 2\omega a \lambda \cos \omega\tau_0)}{1 + (2a\omega\lambda)^2} + \frac{x_0^{\cdot} \rho (\lambda \sin \theta\delta - \delta \cos \theta\delta + \delta e^{-\theta\lambda})}{\delta (\operatorname{ch} \theta\lambda - \cos \theta\delta)}$$

$$\rho = \frac{1+R}{2R}, \qquad \theta = \frac{2\pi n}{\omega}$$

Eliminating the coordinate τ_0 from (3.1), we obtain the equation in $x_0^{\cdot}$

$$\left(d-\frac{x_0^{\cdot}\rho\sin\theta\delta}{\delta(\operatorname{ch}\theta\lambda-\cos\theta\delta)}\right)^2+\left(\frac{x_0^{\cdot}}{\omega}\right)^2\left(\rho\,\frac{e^{-\theta\lambda}-\cos\theta\delta-\dfrac{\lambda}{\delta}\sin\theta\delta}{\operatorname{ch}\theta\lambda-\cos\theta\delta}-1\right)^2=d^2_* \qquad (3.2)$$

A multiple root of Eq. (3.2) corresponds to the boundary of the merging of the stable and the unstable modes. Thus, equating the discriminant of Eq. (3.2) to zero, we obtain the sought boundary surface in parameter space

$$\left(\frac{d_0}{d_*}\right)^2=1+\frac{\omega^2}{\delta^2}\left(\frac{\sin\theta\delta}{e^{\theta\lambda}-\cos\theta\delta-(\lambda/\delta)\sin\theta\delta-\rho^{-1}(\operatorname{ch}\theta\lambda-\cos\theta\delta)}\right)^2 \qquad (3.3)$$

Here, depending on the choice of the point transformation, it is further necessary to satisfy the inequality $x_0^{\cdot}<0$ which in the considered case, reduces to the condition $\sin\theta\delta<0$. The expression

$$\left(\frac{d_0}{d_*}\right)^2=1+\frac{(2\pi n\sin\theta\delta)^2}{\theta^2(\delta\operatorname{sh}\theta\lambda-\lambda\sin\theta\delta)^2},\quad \sin\theta\delta<0 \qquad (3.4)$$

determines the minimum admissible clearance, for the limit value of the coefficient of restitution $R=1$.

The described calculation of d_0 is in some sense an instructive one. The point is that the partitioning of the parameter plane into regions of subharmonic modes is defined by a structure which becomes finer as the order n of the mode increases. At the same time, as follows from (3.4), the ratio of the amplitude of the dangerous subharmonic mode to the amplitude of the forced linear oscillations of the system, corresponding to the same values of the parameters, considerably increases with the increase of the order of the subharmonic mode. The absolute value of the amplitude decreases rapidly with the growth of n, if the excitation level is independent of the frequency and increases with the growth of n, if this level is proportional to the square of the excitation frequency (the case of kinematic excitation). Consequently, the mode can manifest itself the stronger, the finer is the partitioning structure of the parameter space. Figure 5 shows the function $d_0(\omega)$ for $\lambda=0{,}1$, computed from Eq. (3.4).

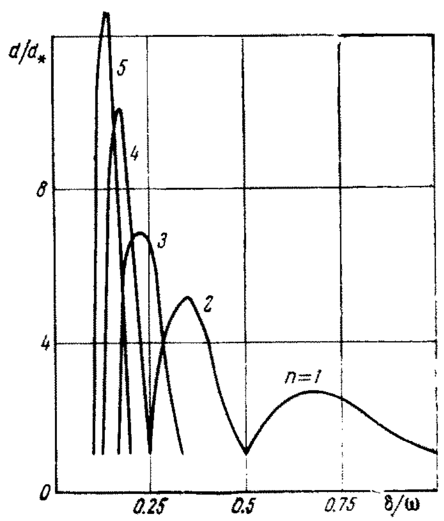

Fig. 5

It should be noted that the pattern of bifurcation considered above is in a sense the simplest one; the regions of existence of the unstable modes $\Gamma(n,1)$ extend from the boundary of generation $d=d_*$ unto the boundary of vanishing by merging with stable modes of the same type. In the general case the mode defined at the instant of generation as $\Gamma(n,k)$, may undergo qualitative changes as d grows. Thus, a more complete investigation of the unstable periodic motions $\Gamma(n,2)$, considered in Example 2, shows the absence of the boundary of their merging with stable modes of precisely the same type.

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Stationary two-dimensional electric current distributions in an anisotropically conducting medium having a nonlinear Ohm's law, are described by the system of equations formulated in [1]. Depending on the character of the nonlinear relation between the current density and the electric field, and on the value of the Hall parameter β, this system can be of an elliptic or hyperbolic type. For $\beta = 0$ the electrodynamic equations are analogous to the equations for potential gas dynamic flows, therefore by analogy these problems can be solved by the hodograph transformation, as it is done in gas dynamics [2]. The hodograph transformation generalized for the case $\beta \neq 0$ is applied below to simple two-dimensional problems. The relation between the type of system and the positive definiteness of the symmetric part of the differential conductivity tensor, is established. Linear equations in the hodograph plane of an effective electric field are obtained for the potential and for a function of the electric current. Boundary conditions are formulated in terms of each of these functions on the image lines for the electrode and dielectric regions with straight-line boundaries. For the elliptic case the solution of two asymptotic problems are obtained and examined: (1) the field distribution in a strip between a perfectly conducting wall and a dielectric wall; (2) the current concentration in the region of a semi-infinite electrode edge. The possibility of corresponding solutions for the hyperbolic case is discussed. For $\beta \neq 0$ exact solutions for particular depen-